

IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING FROM LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study linear differential equations arising from Bessel polynomials and their applications. From these linear differential equations, we give some new and explicit identities for Bessel polynomials.

1. INTRODUCTION

As is well known, the Bessel differential equation is given by

$$(1.1) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2) y = 0, \quad (\text{see [17]}).$$

for an arbitrary complex number α .

The Bessel functions of the first kind $J_\alpha(x)$ are defined by the solution of (1.1).

For $n \in \mathbb{Z}$, $J_n(x)$ are sometimes also called cylinder function or cylindrical harmonics.

It is known that

$$(1.2) \quad J_n(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!} \left(\frac{x}{2}\right)^{2l+n}, \quad (\text{see [1, 16, 17]}).$$

The generating function of Bessel functions is given by

$$(1.3) \quad e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n,$$

and $J_n(x)$ can be also represented by the contour integral as

$$(1.4) \quad J_n(x) = \frac{1}{2\pi i} \oint e^{\frac{x}{2}(t-\frac{1}{t})} t^{-n-1} dt, \quad (\text{see [17]}),$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

The Bessel polynomials are defined by the solution of the differential equation

$$(1.5) \quad x^2 \frac{d^2 y}{dx^2} + 2(x+1) \frac{dy}{dx} - n(n+1)y = 0, \quad (\text{see [1-6, 15, 16]}).$$

Indeed, the solutions of (1.5) are given by

$$(1.6) \quad \begin{aligned} y_n(x) &= \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k \\ &= \sqrt{\frac{2}{\pi x}} e^{\frac{1}{x}} K_{-n-\frac{1}{2}}\left(\frac{1}{x}\right), \quad (\text{see [1, 15-17]}), \end{aligned}$$

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where

$$K_\nu(z) = \frac{\Gamma(\nu + \frac{1}{2})(2z)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2 + z^2)^{\nu + \frac{1}{2}}} dt.$$

We note that $y_n(x)$ are very similar to the modified spherical Bessel function of the second kind.

The first few are given as

$$\begin{aligned} y_0(x) &= 1, & y_1(x) &= x + 1, & y_2(x) &= 3x^2 + 3x + 1, \\ y_3(x) &= 15x^3 + 15x^2 + 6x + 1, \\ y_4(x) &= 105x^4 + 105x^3 + 45x^2 + 10x + 1, & \dots \end{aligned}$$

Carlitz reverse Bessel polynomials are defined by

$$(1.7) \quad p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right), \quad (n \in \mathbb{N} \cup \{0\}), \quad (\text{see } [4, 15]).$$

These polynomials are also given by the generating function as

$$(1.8) \quad e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}.$$

The explicit formulas for them are

$$\begin{aligned} (1.9) \quad p_n(x) &= \sum_{k=1}^n \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} x^k \\ &= (2n-3)!! x {}_1F_1(1-n; 2-2n; 2x), \quad (\text{see } [1, 15, 16]), \end{aligned}$$

where

$$n!! = \begin{cases} n(n-2) \cdots 5 \cdot 3 \cdot 1 & \text{if } n > 0 \text{ odd,} \\ n(n-2) \cdots 6 \cdot 4 \cdot 2 & \text{if } n > 0 \text{ even,} \\ 1 & \text{if } n = -1, 0, \end{cases}$$

and

$$\begin{aligned} {}_1F_1(a; b; z) &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{a(a+1) \cdots (a+k-1)}{b(b+1) \cdots (b+k-1)} \frac{z^k}{k!} \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \end{aligned}$$

The first few polynomials are

$$\begin{aligned} p_1(x) &= x, \\ p_2(x) &= x^2 + x, \\ p_3(x) &= x^3 + 3x^2 + 3x, \\ p_4(x) &= x^4 + 6x^3 + 15x^2 + 15x, \dots \end{aligned}$$

Recently, several authors have studied non-linear differential equations related to special polynomials (see [7–14]).

The reverse Bessel polynomials are used in the design of Bessel electronic filters.

In this paper, we consider linear differential equations arising from Carlitz reverse Bessel polynomials and give some new and explicit identities for Bessel polynomials.

2. IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING FROM LINEAR DIFFERENTIAL EQUATIONS

Let us put

$$(2.1) \quad F = F(t, x) = e^{x(1-\sqrt{1-2t})}.$$

Thus, by (2.1), we get

$$(2.2) \quad F^{(1)} = \frac{d}{dt} F(t, x) = x(1-2t)^{-\frac{1}{2}} F,$$

$$(2.3) \quad \begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \left(x(1-2t)^{-\frac{3}{2}} + x^2(1-2t)^{-1} \right) F, \end{aligned}$$

$$(2.4) \quad \begin{aligned} F^{(3)} &= \frac{d}{dt} F^{(2)} \\ &= \left(3x(1-2t)^{-\frac{5}{2}} + 3x^2(1-2t)^{-2} + x^3(1-2t)^{-\frac{3}{2}} \right) F, \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} F^{(4)} &= \frac{dF^{(3)}}{dt} \\ &= \left(15x(1-2t)^{-\frac{7}{2}} + 15x^2(1-2t)^{-3} + 6x^3(1-2t)^{-\frac{5}{2}} + x^4(1-2t)^{-2} \right) F. \end{aligned}$$

Continuing this process, we set

$$(2.6) \quad \begin{aligned} F^{(N)} &= \left(\frac{d}{dt} \right)^N F(t, x) \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N, x) (1-2t)^{-\frac{i}{2}} \right) F, \end{aligned}$$

where $N = 1, 2, 3, \dots$

From (2.6), we note that

$$(2.7) \quad \begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt} F^{(N)} \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N, x) \left(-\frac{i}{2} \right) (1-2t)^{-\frac{i}{2}-1} (-2) \right) F \\ &\quad + \sum_{i=N}^{2N-1} a_{i-N}(N, x) (1-2t)^{-\frac{i}{2}} F^{(1)} \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N, x) (1-2t)^{-\frac{i+2}{2}} \right) F \\
&\quad + \left(\sum_{i=N}^{2N-1} a_{i-N}(N, x) (1-2t)^{-\frac{i}{2}} \right) x (1-2t)^{-\frac{1}{2}} F \\
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N, x) (1-2t)^{-\frac{i+2}{2}} \right) F + \left(\sum_{i=N}^{2N-1} x a_{i-N}(N, x) (1-2t)^{-\frac{i+1}{2}} \right) F \\
&= \left\{ x a_0(N, x) (1-2t)^{-\frac{N+1}{2}} + (2N-1) a_{N-1}(N, x) (1-2t)^{-\frac{2N+1}{2}} \right. \\
&\quad \left. + \sum_{i=N+1}^{2N-1} ((i-1) a_{i-N-1}(N, x) + x a_{i-N}(N, x)) (1-2t)^{-\frac{i+1}{2}} \right\} F.
\end{aligned}$$

By replacing N by $N+1$ in (2.6), we get

$$\begin{aligned}
(2.8) \quad F^{(N+1)} &= \left(\sum_{i=N+1}^{2N+1} a_{i-N-1}(N+1, x) (1-2t)^{-\frac{i}{2}} \right) F \\
&= \left(\sum_{i=N}^{2N} a_{i-N}(N+1, x) (1-2t)^{-\frac{i+1}{2}} \right) F.
\end{aligned}$$

By comparing the coefficients on both sides (2.7) and (2.8), we have

$$(2.9) \quad a_0(N+1, x) = x a_0(N, x),$$

$$(2.10) \quad a_N(N+1, x) = (2N-1) a_{N-1}(N, x),$$

and

$$(2.11) \quad a_{i-N}(N+1, x) = (i-1) a_{i-N-1}(N, x) + x a_{i-N}(N, x),$$

where $N+1 \leq i \leq 2N-1$.

From (2.2) and (2.6), we can derive the following equation (2.11):

$$(2.12) \quad x (1-2t)^{-\frac{1}{2}} F = F^{(1)} = a_0(1, x) (1-2t)^{-\frac{1}{2}} F.$$

Thus, by (2.12), we have

$$(2.13) \quad a_0(1, x) = x.$$

From (2.9), we note that

$$(2.14) \quad a_0(N+1, x) = x a_0(N, x) = x^2 a_0(N-1, x) = \cdots = x^N a_0(1, x) = x^{N+1},$$

and, by (2.10), we see

$$\begin{aligned}
(2.15) \quad a_N(N+1, x) &= (2N-1) a_{N-1}(N, x) \\
&= (2N-1)(2N-3) a_{N-2}(N-1, x) \\
&\vdots \\
&= (2N-1)(2N-3) \cdots 3 \cdot 1 a_0(1, x) \\
&= (2N-1)!! x.
\end{aligned}$$

The matrix $(a_i(j, x))_{0 \leq i \leq N-1, 1 \leq j \leq N}$ is given by

$$\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\vdots \\
N-1
\end{array}
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & N \\
x & x^2 & x^3 & x^4 & \cdots & x^N \\
& 1!!x & & & & \\
& & 3!!x & & & \\
& & & 5!!x & & \\
& & & & \ddots & \\
& 0 & & & & (2N-3)!!x
\end{bmatrix}$$

From (2.11), we obtain

$$\begin{aligned}
(2.16) \quad & a_1(N+1, x) \\
&= Na_0(N, x) + xa_1(N, x) \\
&= Na_0(N, x) + x(N-1)a_0(N-1, x) + x^2a_1(N-1, x) \\
&\vdots \\
&= \sum_{i=0}^{N-2} x^i (N-i) a_0(N-i, x) + x^{N-1} a_1(2, x) \\
&= \sum_{i=0}^{N-2} x^i (N-i) a_0(N-i, x) + x^{N-1} x \\
&= \sum_{i=0}^{N-1} x^i (N-i) a_0(N-i, x),
\end{aligned}$$

$$\begin{aligned}
(2.17) \quad & a_2(N+1, x) \\
&= (N+1)a_1(N, x) + xa_2(N, x) \\
&= (N+1)a_1(N, x) + xNa_1(N-1, x) + x^2a_2(N-1, x) \\
&\vdots \\
&= \sum_{i=0}^{N-3} x^i (N+1-i) a_1(N-i, x) + x^{N-2} a_2(3, x) \\
&= \sum_{i=0}^{N-3} x^i (N+1-i) a_1(N-i, x) + 3x^{N-2} a_1(2, x) \\
&= \sum_{i=0}^{N-2} x^i (N+1-i) a_1(N-i, x),
\end{aligned}$$

and

$$\begin{aligned}
(2.18) \quad & a_3(N+1, x) \\
&= (N+2)a_2(N, x) + xa_3(N, x) \\
&= (N+2)a_2(N, x) + x(N+1)a_2(N-1, x) + x^2a_3(N-1, x)
\end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = \sum_{i=0}^{N-4} x^i (N-i+2) a_2(N-i, x) + 5x^{N-3} a_2(3, x) \\
& = \sum_{i=0}^{N-3} x^i (N-i+2) a_2(N-i, x).
\end{aligned}$$

Continuing this process, we get

$$(2.19) \quad a_j(N+1, x) = \sum_{i=0}^{N-j} x^i (N-i+j-1) a_{j-1}(N-i, x),$$

where $j = 1, 2, \dots, N-1$.

Now, we give explicit expressions for $a_j(N+1, x)$ ($j = 1, 2, \dots, N-1$). From (2.14) and (2.16), we can easily derive the following equation:

$$\begin{aligned}
(2.20) \quad a_1(N+1, x) &= \sum_{i_1=0}^{N-1} x^{i_1} (N-i_1) a_0(N-i_1, x) \\
&= x^N \sum_{i_1=0}^{N-1} (N-i_1).
\end{aligned}$$

By (2.17), (2.18) and (2.19), we get

$$\begin{aligned}
(2.21) \quad a_2(N+1, x) &= \sum_{i_2=0}^{N-2} x^{i_2} (N-i_2+1) a_1(N-i_2, x) \\
&= x^{N-1} \sum_{i_2=0}^{N-2} \sum_{i_1=0}^{N-2-i_2} (N-i_2+1) (N-i_2-i_1-1),
\end{aligned}$$

$$\begin{aligned}
(2.22) \quad a_3(N+1, x) &= \sum_{i_3=0}^{N-3} x^{i_3} (N-i_3+2) a_2(N-i_3, x) \\
&= x^{N-2} \sum_{i_3=0}^{N-3} \sum_{i_2=0}^{N-3-i_3} \sum_{i_1=0}^{N-3-i_3-i_2} (N-i_3+2) (N-i_3-i_2) \\
&\quad \times (N-i_3-i_2-i_1-2),
\end{aligned}$$

and

$$\begin{aligned}
(2.23) \quad a_4(N+1, x) &= \sum_{i_4=0}^{N-4} x^{i_4} (N-i_4+3) a_3(N-i_4, x) \\
&= x^{N-3} \sum_{i_4=0}^{N-4} \sum_{i_3=0}^{N-4-i_4} \\
&\quad \times \sum_{i_2=0}^{N-4-i_4-i_3} \sum_{i_1=0}^{N-4-i_4-i_3-i_2} (N-i_4+3) (N-i_4-i_3+1) \\
&\quad \times (N-i_4-i_3-i_2-1) (N-i_4-i_3-i_2-i_1-3).
\end{aligned}$$

Continuing this process, we get
(2.24)

$$a_j(N+1, x) = x^{N-j+1} \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} \prod_{k=1}^j (N-i_j-\cdots-i_k-(j-(2k-1))).$$

Therefore, we obtain the following theorem.

Theorem 1. For $N \in \mathbb{N}$, the linear differential equations

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t, x) = \left(\sum_{i=N}^{2N-1} a_{i-N}(N, x) (1-2t)^{-\frac{i}{2}} \right) F$$

has a solution $F = F(t, x) = e^{x(1-\sqrt{1-2t})}$, where

$$\begin{aligned} a_0(N, x) &= x^N, \quad a_{N-1}(N, x) = (2n-3)!!x, \\ a_j(N, x) &= x^{N-j} \sum_{i_j=0}^{N-j-1} \sum_{i_{j-1}=0}^{N-j-1-i_j} \cdots \sum_{i_1=0}^{N-j-1-i_j-\cdots-i_2} \\ &\quad \times \left(\prod_{k=1}^j (N-i_j-i_{j-1}-\cdots-i_k-(j-(2k-2))) \right). \end{aligned}$$

Recall the the reverse Bessel polynomials $p_k(x)$ are given by the generating function as

$$\begin{aligned} (2.25) \quad F &= F(t, x) = e^{x(1-\sqrt{1-2t})} \\ &= \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!}. \end{aligned}$$

Thus, by (2.25), we get

$$\begin{aligned} (2.26) \quad F^{(N)} &= \left(\frac{d}{dt} \right)^N F(t, x) \\ &= \sum_{k=N}^{\infty} p_k(x) (k)_N \frac{t^{k-N}}{k!} \\ &= \sum_{k=0}^{\infty} p_{k+N}(x) (k+N)_N \frac{t^k}{(k+N)!} \\ &= \sum_{k=0}^{\infty} p_{k+N}(x) \frac{t^k}{k!}. \end{aligned}$$

On the other hand, by Theorem 1, we get

$$\begin{aligned} (2.27) \quad F^{(N)} &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N, x) (1-2t)^{-\frac{i}{2}} \right) F \\ &= \sum_{i=N}^{2N-1} a_{i-N}(N, x) \left(\sum_{l=0}^{\infty} \left(-\frac{i}{2} \right)_l \frac{(-2t)^l}{l!} \right) \left(\sum_{m=0}^{\infty} p_m(x) \frac{t^m}{m!} \right) \end{aligned}$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{i=N}^{2N-1} a_{i-N}(N, x) \sum_{l=0}^k \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x) \right\} \frac{t^k}{k!}.$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $k \in \mathbb{N} \cup \{0\}$, and $N \in \mathbb{N}$, we have*

$$p_{k+N}(x) = \sum_{i=N}^{2N-1} a_{i-N}(N, x) \sum_{l=0}^k \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x),$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

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